J Stat Phys (2010) 138: 701–727
DOI 10.1007/s10955-009-9894-y

The Global Statistics of Return Times: Return Time Dimensions Versus Generalized Measure Dimensions

Giorgio Mantica

Received: 22 July 2009 / Accepted: 23 November 2009 / Published online: 5 December 2009

Abstract We investigate return times in dynamical systems, i.e. the time required by a trajectory to complete a return journey to a neighborhood of the initial position. In particular, we study the relations holding between the scaling exponents of phase-space moments of return times in balls of diminishing radius, on the one side, and the generalized dimensions of invariant measures, on the other. Because of a heuristic use of Kac theorem, the former have been used in place of the latter in numerical and experimental investigations: to mark the distinction, we call them return time dimensions. We derive a full set of inequalities linking generalized dimensions of invariant measures and return time dimensions. We comment on their optimality with the aid of two maps due to von Neumann–Kakutani and to Gaspard–Wang. We conjecture a formula for the return time dimensions in a typical system. We only assume that the dynamical system under investigation is ergodic and that motion takes place in a compact, finite dimensional space.

Keywords Renyi spectrum · Hentschel-Procaccia dimensions · Return times · Return time dimensions · Recurrence

1 Introduction: Statement of the Problem and Previous Results

The metric theory of dynamical systems is based on the study of a transformation T of a space X into itself, that preserves a probability measures μ on a suitable sigma algebra $\mathcal{A}$. We assume throughout this paper that the dynamical system $(X, T, \mathcal{A}, \mu)$ under investigation

G. Mantica (✉)
Center for Nonlinear and Complex Systems, Dipartimento di Fisica e Matematica,
Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy
e-mail: giorgio@uninsubria.it

G. Mantica
INFN, Sezione di Milano, Milano, Italy

G. Mantica
CNISM, Unità di Como, Como, Italy

is ergodic and that X is a compact metric space enclosed in $\mathbf{R}^n$. This case is general enough to cover many practical applications. From a physical point of view, μ may be thought of as the invariant distribution, in the space X, of points of a typical trajectory of the system, generated by repeated applications of the map T on a starting point x.

Our interest lies in *return times* of the motion. Let $A \in \mathcal{A}$ be a measurable subset of X of positive measure. Later on, we shall choose A to be a ball, that is, a circular neighborhood of a point. Let x be any point in A. We denote by $\tau_A(x)$ the (integer) time of the first return of x to the set A:

$$\tau_A(x) = \inf\{n > 0 \text{ s.t. } T^n(x) \in A\}. \tag{1}$$

Return times and invariant measures are linked by a variety of results that stand on the pillars of the classical theorems of Poincaré and Kac [29]. The first guarantees that the return time of a point x to the set A is finite, almost surely with respect to any invariant measure μ; the second links the average time needed for recurrence of points in the set A to the inverse of the measure of A. On these bases, it was conjectured long ago by Grassberger [11] and independently by Jensen et al. [21] that the statistical moments of return times, when averaged over balls of radius ε, centered at *all* points of a typical trajectory (therefore, not uniquely fixed as in Poincaré and Kac theorems), have a power-law scaling behavior, when ε tends to zero, with exponents proportional to the generalized dimensions of the measure μ.

Generalized dimensions of measures, defined *à la* Hentschel–Procaccia [2, 3, 8, 11, 12, 19, 34], have a large importance in dynamical systems, see Pesin [27] for a comprehensive review. Their computation is a task of practical and theoretical relevance, for which many alternative techniques have been proposed. Therefore, Grassberger and Jensen et al. idea offers a most interesting alternative in this respect.

Indeed, the original conjecture of has become implicit usage in successive investigations, that have computed generalized dimensions from the statistics of return times. Yet, even before the most recent applications and extensions of this technique [13, 15], this approach has been critically examined in [23]. Stimulated by these findings, we have tried to answer a fundamental question that has frequently been overlooked: whether the conjecture is rigorous and whether it is exact in certain cases, the former obviously implying the latter. In order to disambiguate this point, in this paper we shall call *return time dimensions* the values obtained from the scaling exponent of averages of return times, and we shall investigate whether they are equal to *measure generalized dimensions*.

Before getting into details, observe that the conjecture is bold: generalized measure dimensions are defined independently of the dynamics, while return times obviously are. Put in another way, the same measure (characterized by a spectrum of Hentschel–Procaccia–Pesin generalized dimensions) can be the equilibrium measure of quite different dynamical systems. Precisely because of this, studying the relations holding among the two sets of dimension is interesting, independently of the validity of the above conjecture, since it leads to "universal" results that hold for all dynamical maps T for which a given measure μ is invariant and ergodic.

In a first paper [18] we have studied this problem for invariant measures supported on attractors of Iterated Function Systems. The scope of this work has been successively enlarged in [6] by the analysis of return (and entrance) times in dynamical cylinders (rather than balls) for Bowen–Gibbs measures. Relying on precise approximations to the *local* statistics of return times obtained in [1], the situation for entrance times (a variant of the approach mentioned above) has been almost completely clarified, while that for return times has been settled only for indices $q < 1$ (see below for definitions and further discussions).

Results concerning return times in cylinders and their fluctuations are numerous: see e.g. [7, 32]. For the class of super-disconnected I.F.S. cylinders and balls are in a strict relation, described in [18]. Yet, in the general case, the problem of return times in balls, rather than cylinders, remains completely open and it is arguably the most relevant to practical and numerical applications.

In this paper we advance the analysis of this problem by proving rigorous bounds holding in full generality between measure dimensions and those obtained via return times. In fact, we do not require any additional property (like e.g. Bowen–Gibbs) on the dynamical system under consideration, other than those listed at the beginning of this section. In the course of this analysis we will also consider the comparison between generalized dimensions and their box versions, commonly used in numerical simulations. We shall introduce new box quantities which will be shown to be optimal, both for measure and for return time dimensions, in the sense that they yield exactly the full spectrum of generalized dimensions. In addition to their rôle in numerical computations, these quantities also offer distinctive advantages in the theoretical analysis. Finally, by analyzing the case of two significant one-dimensional maps, we shall demonstrate the optimality of the derived inequalities and we shall put forward a conjecture on the behavior of return times dimensions in a "typical" case.

On the contrary, we shall not consider the problem of the multifractal decomposition, i.e. whether dimensions are linked to the so-called $f(\alpha)$ spectrum [16, 24, 28]. It must also be underlined the difference of this problem—the *global* statistics of return times—with the much more investigated case of the *local* statistics, that consider the distribution of return times of points in a nested sequence of neighborhoods of a single point: see e.g. [1, 17, 20, 25, 32] and references therein.

2 Definitions, Structure of the Paper and Brief Summary of Results

We start by giving formal definitions of generalized dimensions (a variety of possibilities can be found in the literature). Let $B_\varepsilon(x)$ be the ball of radius ε at x and q a real quantity different from one. The *partition functions* $\Gamma_\mu(\varepsilon,q)$ and $\Gamma_\tau(\varepsilon,q)$ are the integrals

$$\Gamma_\mu(\varepsilon,q) := \int_X [\mu(B_\varepsilon(x))]^{q-1} d\mu(x), \tag{2}$$

$$\Gamma_\tau(\varepsilon,q) := \int_X [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x). \tag{3}$$

If the integrand is not summable, we shall understand that the value of the partition function is infinite. The symmetry between the two definitions is apparent and betrays the idea behind the approach mentioned in the Introduction: the measure of a ball appearing in (2) is replaced in (3) by the inverse of the return time of the point at its center. Remark that the integral in (3) can be computed by a Birkhoff sum over a trajectory [18, 23], as in the original proposals [11, 21]. Remark also that the actual numerical computations for [11] were performed with Birkhoff sums of the kind $\sum_{i,j} \tau^{1-q}_{B_\varepsilon(x_i)}(x_j)$ (P. Grassberger, private communication) and therefore they were estimates of the integral $\int [\tau_{B_\varepsilon(x)}(y)]^{1-q} d\mu(x) d\mu(y)$. This amounts to computing entrance (rather than return) times.

The *generalized dimensions* $D^\pm_\sigma(q)$ are defined via the scaling of partition functions for small ε:

$$\Gamma_\sigma(\varepsilon,q) \sim \varepsilon^{D^\pm_\sigma(q)(q-1)}, \tag{4}$$

where the symbol σ from now on denotes either μ or τ. More precisely, one has that, for $q \neq 1$,

$$D_{\sigma}^{\pm}(q) := \limsup(\inf) \frac{1}{q-1} \frac{\log \Gamma_{\sigma}(\varepsilon, q)}{\log \varepsilon}. \tag{5}$$

For $q = 1$, as usual, slightly different definitions are needed:

$$D_{\sigma}^{\pm}(1) := \limsup(\inf) \frac{\Gamma_{\sigma}^{l}(\varepsilon)}{\log \varepsilon}, \tag{6}$$

where

$$\Gamma_{\mu}^{l}(\varepsilon) := \int_X \log[\mu(B_{\varepsilon}(x))], \tag{7}$$

$$\Gamma_{\tau}^{l}(\varepsilon) := \int_X \log[\tau_{B_{\varepsilon}(x)}^{-1}(x)]. \tag{8}$$

As noted, partition functions may be infinite: in such case we shall also set equal to infinity the corresponding generalized dimensions.

The central question addressed in this paper is the nature of the relations between the two sets of dimensions: are they equal, always or in certain cases at least? Can a set of rigorous inequalities among them be derived?

The results of this paper are organized as follows. In the next section we briefly outline basic properties (monotonicity, convexity) of return time generalized dimensions. Then, we shall find it convenient to introduce a number of additional quantities, that we shall also call dimensions and that are interesting on their own. Some of these dimensions are conventional, some are new. In Sect. 4 we start by defining box dimensions, $\Delta_{\mu}^{\pm}(q)$ and $\Delta_{\tau}^{\pm}(q)$, for measure and return times, respectively, following typical usage in experimental and numerical applications: partition functions are defined in relation to partitions of the space X into cubic boxes, and limits are taken with respect to the finesse of the partition.

In Sect. 5 we review the known relations between box and generalized dimensions of measures and we describe a proposal put forward by Riedi [31] to avoid "pathological" values of box dimensions for negative q. By a modification of his idea we define a new box partition function that offers a definite theoretical advantage over both the original box quantities and Riedi's enhanced box formalism: its scaling yields the generalized dimensions $D_{\mu}^{\pm}(q)$ for all values of q, independently of the particular grid adopted. This is made formal in Theorem 1, that, although not directly related to return times, constitutes one of the main results of this paper.

While the previous results deal with measure dimensions, in Sect. 6 we consider the relations between generalized and box dimensions for return times: Proposition 1 shows that the latter are always larger than, or equal to, the former. Mimicking the procedure developed for measures, we introduce a box quantity that yields exactly the generalized return time dimensions $D_{\tau}^{\pm}(q)$, again for all values of q, independently of the particular grid adopted: this is the content of Theorem 2. The rôle of the new partition function introduced in this section is not restricted to numerical simulation: it will be one of the main tools in the proof of two theorems in Sect. 9.

We then put in relation measure and return time dimensions, according to the theme of this paper. Section 7 introduces a central quantity to this goal: the distribution of return times into a fixed set A in the space X. The zeroth and first moment of this distribution are fixed by Poincaré and Kac theorems. Basic inequalities are derived for the remaining moments:

Lemma 5, that can be seen as a sort of generalized Kac theorem. We stress again that this is obtained in the most general setting.

These results are put at work in Sect. 8: inequalities between measure and return box dimensions are derived for all values of q and equality is found for $q = 0$: Proposition 3 gives full detail.

Section 9 is the heart of the paper. There, we chain together our results in Theorem 3, that presents the most complete set of inequalities—that we have been able to prove in full generality—among generalized and box dimensions, of measures and return times:

Theorem 3 *When the dynamical system $(X, T, \mathcal{A}, \mu)$ is ergodic and X is a compact metric space enclosed in $\mathbf{R}^n$, for any grid $\theta \in \Theta$, the dimensions defined in this work are linked by the inequalities: for $q > 0$ one has that $D_\tau^\pm(q) \le \Delta_\tau^\pm(\theta, q) \le \Delta_\mu^\pm(q) = D_\mu^\pm(q)$, while for $q \le 0$ one finds $\Delta_\tau^\pm(\theta, q) \ge \Delta_\mu^\pm(\theta, q) \ge D_\mu^\pm(q)$ and $\Delta_\tau^\pm(\theta, q) \ge D_\tau^\pm(q)$. Finally, for $q = 0$, the equality $\Delta_\tau^\pm(\theta, 0) = \Delta_\mu^\pm(\theta, 0)$ holds.*

In the same section, we discuss the optimality of the inequalities presented. We study the role of short returns, that imply an upper bound for positive dimensions, described in Lemma 7. We also investigate the different situations occurring for positive and negative values of q and we compute return time dimensions in two interesting cases: in full detail for the von Neumann–Kakutani map [35] (Theorem 4) and, partly, for the Gaspard–Wang intermittent map [9] (Theorem 5). Also in this section we formulate a conjecture on the typical behavior of return time generalized dimensions that links it significantly to measure generalized dimensions.

Conclusions are presented briefly in Sect. 10, while three additional sections, 11, 12 and 13 contain the details of the calculations and proofs for the two maps quoted above, as well as side results of some interest.

3 General Properties of Return Time Dimensions

Because of the formal similarity between (2) and (3) some of the properties of generalized measure dimensions also characterize return time dimensions. A couple of these are contained in the following lemma.

Lemma 1 *The return time generalized dimensions $D_\tau^\pm(q)$ are monotone non increasing functions of the index q, and the functions $(q-1)D_\tau^-(q)$ for $q > 1$ and $(q-1)D_\tau^+(q)$ for $q < 1$ are convex.*

Proof Observe that both $\Gamma_\mu(\varepsilon, q)$ and $\Gamma_\tau(\varepsilon, q)$ can be seen as integral of a function $\phi(x)$ raised to the power $q - 1$. In the return time case this latter is $\phi(x) = 1/\tau_{B_\varepsilon(x)}(x)$. The two results above are then a consequence of Jensen and Holder inequalities, similar to those holding for $D_\mu^\pm(q)$, whose details can be found in [4, 33]. □

Additional results can be obtained in this line, but will be reported elsewhere. In fact, our specific aim in this paper is simply to compare the value of measure and return time dimensions. In this regard, finiteness of the return time dimensions is an important issue that will be considered in Sects. 6 and 9.

4 Box Dimensions of Measures and of Return Times Distributions

Usually, in numerical experiments, rather than computing the integral (2) one covers the set $X \subset \mathbf{R}^n$ by a lattice of hypercubic boxes A_j, $j = 0, 1, \dots$ of side ε. The usual choice is to draw the zeroth box as having the origin of the coordinates as a corner and the sides exiting from that corner oriented as the coordinate directions. Clearly, different choices are possible, varying origin and orientation. We shall let $\theta \in \Theta$ denote the particular choice of origin and orientation in the set Θ of all choices. θ will be called a *grid*. Therefore, a grid consists of infinitely many box partitions of X, one for every value of ε.

We then consider in place of the partition functions $\Gamma_\mu(\varepsilon, q)$, the sums

$$\Upsilon_\mu(\theta, \varepsilon, q) := \sum_{j \text{ s.t. } \mu(A_j)>0} \mu(A_j)^q. \tag{9}$$

For simplicity of notation, the dependence of A_j on θ and ε will be left implicit here and in the following.

Similarly, by replacing in (3) the box centered at x with the set A_j that contains x, we define the *return time box partition function*:

$$\Upsilon_\tau(\theta, \varepsilon, q) := \sum_j \int_{A_j} \tau_{A_j}^{1-q}(x) d\mu(x). \tag{10}$$

It has to be noticed the double role of the set A_j, as starting and arrival set of the motion. We shall break this symmetry later on. The logarithmic analogues are

$$\Upsilon_\mu^l(\theta, \varepsilon) := \sum_{j \text{ s.t. } \mu(A_j)>0} \mu(A_j) \log[\mu(A_j)], \tag{11}$$

and

$$\Upsilon_\tau^l(\theta, \varepsilon) := \sum_j \int_{A_j} \log[\tau_{A_j}^{-1}(x)] d\mu(x). \tag{12}$$

Define now the *box generalized dimensions* $\Delta_\mu^\pm(\theta, q)$ and $\Delta_\tau^\pm(\theta, q)$, by using Υ's and Δ's in place of Γ's and D's, respectively, in (5) and (6):

$$\Delta_\sigma^\pm(\theta, q) := \limsup(\inf) \frac{1}{q-1} \frac{\log \Upsilon_\sigma(\theta, \varepsilon, q)}{\log \varepsilon}, \tag{13}$$

for $q \neq 1$, while for $q = 1$:

$$\Delta_\sigma^\pm(\theta, 1) := \limsup(\inf) \frac{\Upsilon_\sigma^l(\theta, \varepsilon)}{\log \varepsilon}, \tag{14}$$

where σ can be either μ or τ.

Notice that it is possible to avoid dependence on the specific grid by taking the *infimum* (or the supremum, according to the value of q) over all partitions in the definition of the function $\Upsilon_\tau(\varepsilon, q)$ [27]. We elect not to take this step for two reasons. The first is that this is difficultly achievable in numerical applications and therefore forcing it into the theory does not provide a good model of what is numerically observed. The second is that we shall strive at obtaining results that do not depend on the particular grid selected, but apply to all and *a fortiori* also to dimensions defined with the infimum procedure included. A first instance of this fact is to be met in the next section, where we introduce enhanced box dimensions, based on an idea pioneered by Riedi.

5 Box Versus Generalized Measure Dimensions

The relations between box and generalized measure dimensions is a subject that has been intensively studied, see e.g. [4, 10, 14, 27, 28, 31], with an effort towards proving their equivalence, on the one side, and towards releasing the request of performing an infimum over partitions, on the other side. We first review the known relations needed for our scope in the following lemma, and then we present a new result that we believe to be of some importance.

Lemma 2 *The following relations exists between box and generalized measure dimensions*:

$$\begin{aligned} \Delta_\mu^\pm(\theta, q) &= D_\mu^\pm(q) \quad \textit{for } q > 0, \\ \Delta_\mu^\pm(\theta, q) &\geq D_\mu^\pm(q) \quad \textit{for } q \leq 0. \end{aligned} \tag{15}$$

Proof The full proof, including the non-trivial interval $q \in (0, 1]$, can be found in the complete exposition [4]. Notice that no infimum procedure over ε-grids is involved. □

Therefore, measure box dimensions are independent of the choice of the grid θ for any $q > 0$. Examples exist showing both such dependence and strict inequality w.r.t. generalized dimensions for $q < 0$. The case $q = 0$ seems to be on less firm ground, see Sect. 9. Roughly speaking, what might happen for negative q is the following: if a box A_j "barely touches" the support of the measure μ close to one of its edges, its measure can be arbitrarily small, independently of its size ε, so that $\Delta_\mu^+(\theta, q)$ can become arbitrarily large. To avoid this effect Riedi [31] introduced the following sums—compare (9):

$$\Phi_\mu(\theta, \varepsilon, q) := \sum_{j \text{ s.t. } \mu(A_j) > 0} \mu(\overline{A}_j)^q, \tag{16}$$

where $\overline{A}_j$ is a box of side 3ε centered on the box A_j. In one dimension, for instance, $\overline{A}_j$ consists of the union of A_{j-1}, A_j and A_{j+1}. The geometrical situation in two and more dimensions can be easily pictured by the reader.

Using clever manipulations, Riedi has been able to prove that the dimensions generated by the scaling of Φ_μ coincide with $D_\mu^\pm(q)$ for $q > 1$. It actually follows from estimates in [4] that equality can be proven for any $q > 0$. Although it is plausible that this also holds in large generality for negative q as well (see the numerical results in [26]), we have not been able to find a formal proof of this fact, that would hold in the most general setting adopted in this paper. Yet, in this endeavor, we have discovered a new box quantity that achieves this goal:

Theorem 1 *For any $\theta \in \Theta$, the scaling behavior of the partition functions defined via*:

$$\Psi_\mu(\theta, \varepsilon, q) := \sum_{j \text{ s.t. } \mu(A_j) > 0} \mu(A_j)\mu(\overline{A}_j)^{q-1}, \tag{17}$$

for all $q \neq 1$ and

$$\Psi^l_\mu(\theta,\varepsilon) := \sum_{j \text{ s.t. } \mu(A_j)>0} \mu(A_j)\log[\mu(\overline{A}_j)], \tag{18}$$

for $q=1$ yields the generalized dimensions $D^\pm_\mu(q)$.

Proof of Theorem 1 is a direct consequence of the auxiliary results collected in:

Lemma 3 *For any $\theta \in \Theta$, the following inequalities hold*:

$$\Gamma_\mu(\varepsilon,q) \le \Psi_\mu(\theta,\varepsilon,q) \le \Phi_\mu(\theta,\varepsilon,q), \quad q \ge 1; \tag{19}$$

$$\Gamma_\mu(\varepsilon,q) \ge \Psi_\mu(\theta,\varepsilon,q) \le \Phi_\mu(\theta,\varepsilon,q), \quad q \le 1; \tag{20}$$

$$\Gamma_\mu(k\varepsilon,q) \le \Psi_\mu(\theta,\varepsilon,q), \quad q \le 1; \tag{21}$$

$$\Gamma_\mu(k\varepsilon,q) \ge \Psi_\mu(\theta,\varepsilon,q), \quad q \ge 1; \tag{22}$$

$$\Gamma^l_\mu(k\varepsilon) \ge \Psi^l_\mu(\theta,\varepsilon) \ge \Gamma^l_\mu(\varepsilon). \tag{23}$$

In the above, k is a fixed multiplier that depends only on the Euclidean dimension of the space X.

Proof To prove the first inequality we follow [31]. Since

$$\Gamma_\mu(\varepsilon,q) := \int_X d\mu(x)[\mu(B_\varepsilon(x))]^{q-1} = \sum_j \int_{A_j} d\mu(x)[\mu(B_\varepsilon(x))]^{q-1} \tag{24}$$

and since $B_\varepsilon(x) \subset \overline{A}_j$ when $x \in A_j$, $q \ge 1$,

$$\Gamma_\mu(\varepsilon,q) \le \sum_j \int_{A_j} d\mu(x)[\mu(\overline{A}_j)]^{q-1} = \Psi_\mu(\theta,\varepsilon,q) \le \Phi_\mu(\theta,\varepsilon,q). \tag{25}$$

Equally,

$$\Gamma^l_\mu(\varepsilon) \le \sum_j \int_{A_j} d\mu(x)\log[\mu(\overline{A}_j)] = \Psi^l_\mu(\theta,\varepsilon). \tag{26}$$

Conversely, when $q \le 1$, the first inequality in (25) is reversed, while the second still holds:

$$\Gamma_\mu(\varepsilon,q) \ge \sum_j \int_{A_j} d\mu(x)[\mu(\overline{A}_j)]^{q-1} = \Psi_\mu(\theta,\varepsilon,q) \le \Phi_\mu(\theta,\varepsilon,q). \tag{27}$$

Next, we use the fact that $\overline{A}_j \subset B_{k\varepsilon}(x)$ when $x \in A_j$ and k is a fixed multiplier, as in Sect. 6, to obtain, still for $q \le 1$

$$\begin{aligned}\Gamma_\mu(k\varepsilon,q) &= \sum_j \int_{A_j} [\mu(B_{k\varepsilon}(x)]^{q-1} d\mu(x) \\ &\le \sum_j \int_{A_j} [\mu(\overline{A}_j)]^{q-1} d\mu(x) = \sum_j \mu(A_j)[\mu(\overline{A}_j)]^{q-1} = \Psi_\mu(\theta,\varepsilon,q). \end{aligned} \tag{28}$$

Finally, for $q \geq 1$, we get

$$\varGamma_\mu(k\varepsilon, q) \geq \sum_j \int_{A_j} [\mu(\overline{A}_j)]^{q-1} d\mu(x) = \varPsi_\mu(\theta, \varepsilon, q), \tag{29}$$

$$\varGamma_\mu^l(k\varepsilon) \geq \sum_j \int_{A_j} d\mu(x) \log[\mu(\overline{A}_j)] = \varPsi_\mu^l(\theta, \varepsilon). \tag{30}$$

□

Theorem 1 asserts that the grid-dependent sums $\varPsi_\mu(\theta, \varepsilon, q)$ give rise to a set of dimensions that are independent of the grid θ and coincide with the generalized dimensions $D_\mu^\pm(q)$ for all values of q. Observe that, from a numerical point of view, the sums $\varPsi_\mu(\theta, \varepsilon, q)$ can be evaluated with the same effort required for computing the original sums (9) or Riedi's extension (16): all one needs to know is the value of the box measures $\mu(A_j)$. We therefore believe that Theorem 1 can become a new tool in the multifractal analysis of measures.

6 Box Versus Generalized Return Time Dimensions

In a symmetrical way to what done in the previous section for measure dimensions, we now compare box and generalized return time dimensions, $\varDelta_\tau^\pm(\theta, q)$ and $D_\tau^\pm(q)$. Then, we introduce a new box partition functions for return times, $\varPsi_\tau$, analogous to the $\varPsi_\mu$ of the previous section. We prove that this box partition function yields the generalized return time dimensions $D_\tau^\pm(q)$ for all values of q, independently of the grid θ.

Proposition 1 *The box return time dimensions $\varDelta_\tau^\pm(q)$ are always larger than, or equal to, their generalized counterparts, for all values of q:*

$$\varDelta_\tau^\pm(\theta, q) \geq D_\tau^\pm(q). \tag{31}$$

Proof Fix a specific grid θ and let $j(x)$ be the index of the hypercube of side ε containing the point x. Then, $A_{j(x)}$ is enclosed in the ball of radius $k\varepsilon$ centered at x, with a fixed multiplier $k \geq 1$ that can be chosen as a function only of the (Euclidean) dimension of the space. This implies that $\tau_{B_{k\varepsilon}(x)}(x) \leq \tau_{A_{j(x)}}(x)$ for all x. Therefore, we split the integral defining $\varGamma_\tau(k\varepsilon, q)$ over the partition of side ε,

$$\varGamma_\tau(k\varepsilon, q) = \int_X \tau_{B_{k\varepsilon}(x)}^{1-q}(x) d\mu(x) = \sum_j \int_{A_j} \tau_{B_{k\varepsilon}(x)}^{1-q}(x) d\mu(x), \tag{32}$$

and we use this inequality, first for $1 - q \geq 0$, to get

$$\varGamma_\tau(k\varepsilon, q) \leq \sum_j \int_{A_j} \tau_{A_j}^{1-q}(x) d\mu(x) = \varUpsilon_\tau(\theta, \varepsilon, q). \tag{33}$$

For $1 - q \leq 0$ we obtain the reverse inequality. In force of these inequalities, an immediate calculation provides the thesis. As before, the case $q = 1$ requires a separate treatment:

$$\varGamma_\tau^l(k\varepsilon) = \sum_j \int_{A_j} \log[\tau_{B_{k\varepsilon(x)}}^{-1}(x)] d\mu(x) \geq \sum_j \int_{A_j} \log[\tau_{A_j}^{-1}(x)] d\mu(x) = \varUpsilon_\tau^l(\theta, \varepsilon). \tag{34}$$

Using this information in the limits (5), (6) yields the thesis. □

The estimates in the previous proof help us also to establish existence of the return time partition functions for $q \geq 0$.

Lemma 4 *The partition sums $\Gamma_\tau(\varepsilon, q)$ and $\Upsilon_\tau(\theta, \varepsilon, q)$, as well as $\Upsilon^l_\tau(\theta, \varepsilon)$ and $\Gamma^l_\tau(\varepsilon)$ exist for any $\theta \in \Theta$, $q \geq 0$.*

Proof Existence is trivial for $q > 1$. If $0 \leq q \leq 1$ the functions $\Upsilon_\tau(\theta, \varepsilon, q)$ and $\Upsilon^l_\tau(\theta, \varepsilon)$ exist because of Kac theorem [22]. Then, the inequality (33), valid for $q \leq 1$ and the inequality (34) imply that also $\Gamma_\tau(\varepsilon, q)$ and $\Gamma^l_\tau(\varepsilon)$ exist. □

The ideas exploited in the previous section can also be used to construct a box quantity capable of generating the generalized return time dimensions $D_\tau(q)$. This is defined via

$$\Psi_\tau(\theta, \varepsilon, q) := \sum_j \int_{A_j} \tau^{1-q}_{\overline{A}_j}(x)\, d\mu(x), \tag{35}$$

for $q \neq 1$, and, for $q = 1$ via

$$\Psi^l_\tau(\theta, \varepsilon) := \sum_j \int_{A_j} \log(\tau^{-1}_{\overline{A}_j}(x))\, d\mu(x). \tag{36}$$

Difference with (10) has to be appreciated: the integral is taken over the set A_j, but the return time is computed when x gets back into the larger set $\overline{A}_j$, defined as in Sect. 5.

Theorem 2 *The scaling behavior of the partition functions $\Psi_\tau(\theta, \varepsilon, q)$ for all $q \neq 1$ and $\Psi^l_\tau(\theta, \varepsilon)$ for $q = 1$ yield the generalized dimensions $D^\pm_\tau(q)$ for any $\theta \in \Theta$.*

Proof Let again $j(x)$ be the index of the hypercube of side ε containing the point x. The key point is that

$$B_\varepsilon(x) \subset \overline{A}_{j(x)} \subset B_{k\varepsilon}(x) \tag{37}$$

with a dimension-dependent constant k. Therefore,

$$\tau_{B_\varepsilon(x)}(x) \geq \tau_{\overline{A}_{j(x)}}(x) \geq \tau_{B_{k\varepsilon}(x)}(x), \tag{38}$$

which leads to

$$\Gamma_\tau(\varepsilon, q) = \sum_j \int_{A_j} \tau^{1-q}_{B_\varepsilon(x)}(x) d\mu(x) \leq \Psi_\tau(\theta, \varepsilon, q) \leq \Gamma_\tau(k\varepsilon, q) \tag{39}$$

for $q \geq 1$ and to a reverse chain of inequalities when $q \leq 1$. The logarithmic partition function, to be used for $q = 1$, satisfies

$$\Gamma^l_\tau(\varepsilon) = \sum_j \int_{A_j} \log(\tau^{-1}_{B_\varepsilon(x)}(x)) d\mu(x) \leq \sum_j \int_{A_j} \log(\tau^{-1}_{\overline{A}_{j(x)}}(x)) d\mu(x) = \Psi^l_\tau(\theta, \varepsilon), \tag{40}$$

and

$$\Psi^l_\tau(\theta,\varepsilon) \le \sum_j \int_{A_j} \log(\tau^{-1}_{B_{k\varepsilon}(x)}(x))d\mu(x) = \Gamma^l_\tau(k\varepsilon), \tag{41}$$

from which the thesis follows. □

Remark that the geometric relation in (37) shows that return time dimensions are somehow performing the same kind of action implied in Riedi's enlarged box idea. Also remark that one is free to chose a different grid θ at any value of ε.

The rôle of the partition function $\Psi_\tau(\theta,\varepsilon,q)$ goes well beyond numerical simulations: it will be crucial for the proof of Theorems 4 and 5.

We end this section by showing the existence of a particular combination of $\Upsilon_\tau(\theta,\varepsilon,q)$ that also yields the generalized dimensions for q larger than one. This is defined as follows. Fix a grid θ_0 in $\mathbf{R}^d$ and a value $\varepsilon > 0$. Let e_i, $i = 1,\ldots,d$ unit orthogonal vectors giving the direction of the grid. On this basis, construct 3^d parallel grids with the same directions of θ_0: call them θ_l, $l = 0,\ldots,3^d-1$. The first of these grids is the original θ_0, the others have origins shifted by lattice vectors of the kind $\varepsilon\sum_{i=1}^d n_i e_i$, where the n_i can take the values $0,1,2$. For each of these grids, consider boxes of side 3ε, and on this basis, construct the box partition function

$$\tilde{\Upsilon}_\tau(\theta_0,\varepsilon,q) := \sum_{l=0}^{3^d-1} \Upsilon_\tau(\theta_l,3\varepsilon,q). \tag{42}$$

Proposition 2 *The box partition function $\tilde{\Upsilon}_\tau(\theta_0,3\varepsilon,q)$ yields the generalized dimensions $D^\pm_\tau(q)$ for any $q > 1$, independently of the choice of the grid.*

Proof Let θ_0 be a given grid. Obviously, one has

$$\Psi_\tau(\theta_0,\varepsilon,q) := \sum_j \int_{A_j} \tau^{1-q}_{A_j}(x)\,d\mu(x) \le \sum_j \int_{\overline{A}_j} \tau^{1-q}_{\overline{A}_j}(x)\,d\mu(x), \tag{43}$$

where each integral has been extended to a larger domain. The summation index j runs over all boxes of size ε, while the enlarged boxes $\overline{A}_j$ have side 3ε and each of these is composed of 3^d smaller ones, d being the euclidean space dimension. Neighboring boxes $\overline{A}_j$ overlap, but at the same time one can part the j summation into 3^d different sets of non-overlapping, adjacent boxes. These are precisely defined by the θ_l grids defined above, so that (43) becomes

$$\Psi_\tau(\theta_0,\varepsilon,q) \le \sum_{l=0}^{3^d-1} \Upsilon_\tau(\theta_l,3\varepsilon,q) = \tilde{\Upsilon}_\tau(\theta_0,\varepsilon,q). \tag{44}$$

The above equation is valid for all values of q. Let now $q > 1$. Then, since $\Upsilon_\tau(\theta,\varepsilon,q) \le \Gamma_\tau(k\varepsilon,q)$ for any θ (see (33)) and using also (39), we find

$$\Gamma_\tau(\varepsilon,q) \le \Psi_\tau(\theta_0,\varepsilon,q) \le \sum_{l=0}^{3^d-1} \Upsilon_\tau(\theta_l,3\varepsilon,q) = \tilde{\Upsilon}_\tau(\theta_0,\varepsilon,q) \le 3^d\Gamma_\tau(3k\varepsilon,q). \tag{45}$$

The by-now usual technique proves the thesis. □

7 Kac Theorem and Moment Inequalities

We need to bridge the gap between measure and return time dimensions. To do this, our main tool will be Kac theorem [22], that we put at work in this section. For any measurable set A of positive measure, define the discrete return time measure ν^A via

$$\nu^A(\{j\}) := \mu(\{x \in A \text{ s.t. } \tau_A(x) = j\})/\mu(A). \tag{46}$$

In words, $\nu^A(\{j\})$ is the normalized measure of the set of points of A that return to A in j time steps. Obviously, ν^A is a measure supported on the positive integers, a fact that will be exploited momentarily. Poincaré Theorem guarantees that ν^A is a probability measure:

$$\sum_{j=1}^{\infty} \nu^A(\{j\}) = 1. \tag{47}$$

We shall study the moments of this measure. For $s \in \mathbf{R}$, let ν_s^A be:

$$\nu_s^A := \sum_{j=1}^{\infty} j^s \nu^A(\{j\}) = \frac{1}{\mu(A)} \int_A [\tau_A(x)]^s d\mu(x). \tag{48}$$

Define also the logarithmic moment:

$$\nu_l^A := \sum_{j=2}^{\infty} \log(j) \nu^A(\{j\}). \tag{49}$$

Under the ergodicity hypothesis that we are assuming throughout, Kac theorem fixes the value of the first moment of this measure:

$$\nu_1^A = 1/\mu(A). \tag{50}$$

The key ingredient of our theory is the fact that all moments ν_s^A can be put in relation to the latter, that is to say, to $\mu(A)$. In fact, we have the following lemma.

Lemma 5 *Let ν be a probability measure supported on $[1, \infty)$. Let ν_s be its moments, allowing for an infinite value of these latter. As a function of s, ν_s is monotonic, non-decreasing. Furthermore,*

$$\begin{aligned} \nu_s &\le (\nu_1)^s \quad \textit{for } 0 \le s \le 1, \\ \nu_s &\ge (\nu_1)^s \quad \textit{for } s \le 0, \ \textit{or } s \ge 1. \end{aligned} \tag{51}$$

Proof This lemma is elementary. It follows from a judicious use of Hölder inequality, together with the significant piece of information that the shortest value of return times is one, so that ν is a probability measure supported on $[1, \infty)$. □

Lemma 6 *In the same hypotheses of Lemma 5, one has $\nu_l := \int \log(x) d\nu(x) \le \log(\nu_1)$.*

Proof Since ν is a probability measure, this is Jensen's inequality. □

Because of the observations made at the beginning of this section, the above lemmas apply to ν_s^A, the moments of the return times of points in any positive measure set A, when taken with respect to the normalized measure $d\mu_A(x) = \frac{1}{\mu(A)} d\mu(x)$. As such, the formulae (51) extend the content of Kac theorem to all moments. Later in the paper, we shall find examples where inequalities (51) are strict, as well as examples where they hold as equalities. We shall now investigate the mathematical implications of these results to the dimension problem.

8 Inequalities Between Measure and Return Time Box Dimensions

On the basis of the theory of the previous section, Lemmas 5 and 6, we can now study the quantities $\Upsilon_\sigma(\theta, \varepsilon, q)$ and the associated dimensions $\Delta_\sigma^\pm(\theta, q)$.

Proposition 3 *The box dimensions $\Delta_\sigma^\pm(\theta, q)$, $\sigma = \mu, \tau$, for any $\theta \in \Theta$ are linked by the inequalities*

$$\begin{aligned} &\Delta_\tau^\pm(\theta, q) \geq \Delta_\mu^\pm(\theta, q) \quad \text{for } q < 0, \\ &\Delta_\tau^\pm(\theta, q) \leq \Delta_\mu^\pm(\theta, q) \quad \text{for } q > 0, \\ &\Delta_\tau^\pm(\theta, 0) = \Delta_\mu^\pm(\theta, 0). \end{aligned} \tag{52}$$

Proof Observe that

$$\int_{A_j} \tau_{A_j}^{1-q}(x) d\mu(x) = \mu(A_j) \nu_{1-q}^{A_j} \tag{53}$$

so that

$$\Upsilon_\tau(\theta, \varepsilon, q) = \sum_{j \text{ s.t. } \mu(A_j) > 0} \mu(A_j) \nu_{1-q}^{A_j}. \tag{54}$$

Therefore, using Lemma 5 and Kac theorem, (50), we get

$$\Upsilon_\tau(\theta, \varepsilon, q) \leq \sum_{j \text{ s.t. } \mu(A_j) > 0} \mu(A_j)^q = \Upsilon_\mu(\theta, \varepsilon, q) \tag{55}$$

for $q \in (0, 1)$ and $\Upsilon_\tau(\theta, \varepsilon, q) \geq \Upsilon_\mu(\theta, \varepsilon, q)$ in the opposite case. Using now (2) and (4) we can prove the two inequalities in (52), for $q \neq 1$. The case $q = 1$ can be treated by writing

$$\Upsilon_\tau^l(\theta, \varepsilon) = - \sum_{j \text{ s.t. } \mu(A_j) > 0} \mu(A_j) \nu_l^{A_j}. \tag{56}$$

Using Lemma 6, we arrive at $\Upsilon_\tau^l(\theta, \varepsilon) \geq \Upsilon_\mu^l(\theta, \varepsilon)$ and hence the thesis follows. Finally, direct computation shows that $\Upsilon_\tau(\theta, \varepsilon, 0) = \Upsilon_\mu(\theta, \varepsilon, 0)$ so that $\Delta_\tau^\pm(\theta, 0) = \Delta_\mu^\pm(\theta, 0)$. □

9 All Things Considered: Main Theorems, Comments and Examples

We can now complete our work, first by linking together the inequalities obtained so far and then by commenting on their optimality with the aid of the von Neumann–Kakutani Map [35] and of an intermittent map due to Pomeau–Manneville [30] and Gaspard–Wang [9]. Recall that we have put ourselves in a rather general setting, by requiring only ergodicity of the dynamical system considered. Our fundamental result is therefore:

Theorem 3 *When the dynamical system $(X, T, \mathcal{A}, \mu)$ is ergodic and X is a compact metric space enclosed in $\mathbf{R}^n$, for any $\theta \in \Theta$, the different dimensions defined in this work are linked by the inequalities*:

$$D_\tau^\pm(q) \le \Delta_\tau^\pm(\theta, q) \le \Delta_\mu^\pm(q) = D_\mu^\pm(q), \quad \text{for } q > 0, \tag{57}$$

$$\Delta_\tau^\pm(\theta, q) \ge \Delta_\mu^\pm(\theta, q) \ge D_\mu^\pm(q), \quad \text{for } q \le 0, \tag{58}$$

$$\Delta_\tau^\pm(\theta, q) \ge D_\tau^\pm(q), \quad \text{for } q \le 0, \tag{59}$$

and, for $q = 0$,

$$\Delta_\tau^\pm(\theta, 0) = \Delta_\mu^\pm(\theta, 0). \tag{60}$$

Proof Use Lemma 2 together with Propositions 1 and 3. □

Remarks and comments are now in order.

The only equality that we have proven to hold in full generality is between $\Delta_\tau^\pm(\theta, 0)$ and $\Delta_\mu^\pm(\theta, 0)$, obviously when computed on the same grid θ. It is believed that $\Delta_\mu^\pm(\theta, 0) = D_\mu^\pm(0)$ should hold in large generality [4]. When this is the case, we can also assess that $\Delta_\tau^\pm(\theta, 0)$ does not depend on the grid θ and this provides us with a means of computing the capacity dimensions $D_\mu^\pm(0)$ via return times. It is remarkable that no exceptions to the desired equality have been known until very recently: that is, only a case (still unpublished) is known where $\Delta_\mu^-(\theta, 0)$ is strictly larger than $D_\mu^-(0)$ (S. Tcheremchantsev, private communication).

The situation occurring for $q > 0$ is fully described by a single chain of inequalities, (57). We want now to show that they can be strict. In fact, the return time dimensions $D_\tau^\pm(q)$ may decay to zero when q tends to infinity even when measure dimensions do not. This can be regarded as a consequence of "short returns", a rather general occurrence. In fact, for $m = 1, 2, \ldots$ let

$$\rho(\varepsilon; m) := \mu(\{x \in X \text{ s.t. } \tau_{B_\varepsilon(x)}(x) = m\}) \tag{61}$$

be the distribution of the first return times of a point x into the ball of radius ε centered at x. Observe that this is a global quantity, since x takes values in all of X. Also consider the integrated distribution $R(\varepsilon; k)$:

$$R(\varepsilon; k) := \sum_{m=1}^{k} \rho(\varepsilon; m). \tag{62}$$

We have the following

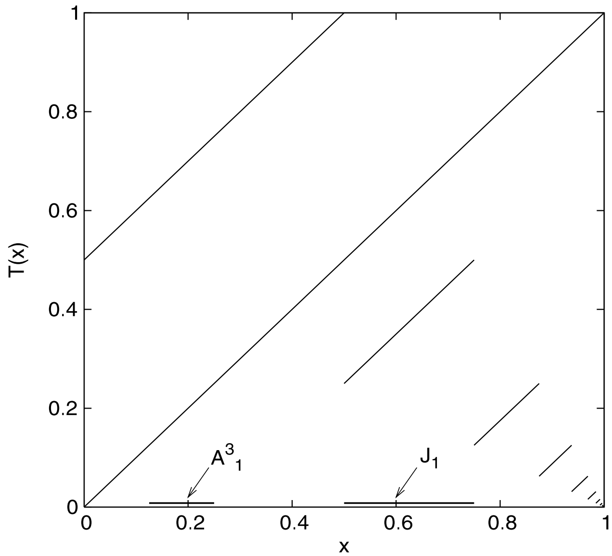
Lemma 7 *If for some $k \ge 1$, there exist constants C and $\delta > 0$ such that $R(\varepsilon; k) \ge C\varepsilon^\delta$, then $D_\tau^\pm(q) \le \frac{\delta}{q-1}$ for all $q > 1$.*

Proof Let $q > 1$. Clearly,

$$\Gamma_\tau(\varepsilon, q) = \sum_{m=1}^{\infty} \rho(\varepsilon; m) m^{1-q} \ge k^{1-q} R(\varepsilon; k) \ge C k^{1-q} \varepsilon^\delta, \tag{63}$$

which yields the thesis. □

Fig. 1 Graph of the von Neumann–Kakutani map defined in (68). Also drawn are the slope-one line through the origin and the sets J_1 and A_1^3



A similar lemma holds obviously also for $\Delta_\tau^\pm(\theta,q)$. This lemma shows that, roughly speaking, in order for dimensions not to tend to zero when q tends to infinity, the probability of small returns must vanish faster than any power of ε, when ε tends to zero. We shall momentarily describe a system, the von Neumann–Kakutani map, where to the contrary this probability decays as ε and the inequalities (57) are strict for $q > 2$.

In a previous work [18] we have outlined another mechanism for short returns: the existence of fixed points of a continuous map T. Indeed, $R(\varepsilon;1)$ can be bounded from below by the measure of a box of radius proportional to ε centered at any fixed point. This latter scales, for small ε, with the local dimension at the fixed point, a value that can be used in Lemma 7. We must remark that in the examples presented in [18] such local dimensions yield an upper bound to the true $D_\tau^\pm(q)$, while the exact asymptotic result should involve the correlation dimension $D_\mu(2)$ (see below).

Let us now consider the case $q < 0$. In full generality, we can only establish the shorter chains of inequalities (58) and (59). We are not able to perform other comparisons. Contrary to what might seem at first blush, this is not the result of a deficiency of our technique. In fact, as we have remarked in Sect. 5, it may happen that $\Delta_\mu^+(\theta,q)$ be larger than $D_\mu^+(q)$, or even infinite, the reason not being any peculiarity of the measure μ, but an unfortunate choice of the grid θ. In turns, this fact also "spoils" $\Delta_\tau^+(\theta,q)$, because of the inequality (58), but *not* $D_\tau^+(q)$, which is *smaller than* $\Delta_\tau^+(\theta,q)$ and, as such, is not linked to $\Delta_\mu^+(\theta,q)$.

This happens precisely for the von Neumann–Kakutani map [35], described in detail in Sect. 11 and pictured in Fig. 1, whose absolutely continuous invariant measure is the uniform Lebesque measure over the unit interval. This map $T:[0,1]\to[0,1]$ is a sort of infinite intervals exchange map that permutes diadic sub-intervals of any order. In this permutation, points in any binary interval of length 2^{-n} (for any integer value of n) "visit once" all remaining intervals before returning home: see Lemma 8 in Sect. 11. From the point of view of return times, this is a sort of dream situation, where all points return in a time $\tau = 2^n$, that is exactly equal to the inverse of the measure of the interval. As a consequence, for these sets, formulae (51) hold as equalities for all real values of s. Nonetheless, Grassberger and Jensen et al. conjecture is verified only partially for this dynamical system, as the following theorem shows:

Theorem 4 *In the dynamical system* $(T, [0, 1], \lambda)$, *where* T *is the von Neumann–Kakutani map defined in* (68) *and* λ *is the Lebesgue measure*, *the dimensions defined in this work take the values*:

$$D_\tau^\pm(q) = \begin{cases} 1 & \text{for } q \le 2, \\ \frac{1}{q-1} & \text{for } q > 2. \end{cases} \tag{64}$$

Moreover, for all grids θ,

$$\Delta_\tau^+(\theta, q) = \Delta_\mu^+(\theta, q) = \infty \quad \text{for } q < 0. \tag{65}$$

Finally, there are an infinite number of grids θ *for which*

$$\begin{aligned} \Delta_\tau^+(\theta, q) &= 1 \quad \text{for } q \ge 0, \\ \Delta_\tau^-(\theta, q) &= 1 \quad \text{for } q \le 0. \end{aligned} \tag{66}$$

Proof See Sect. 12. □

Recall now that for the Lebesgue measure on the unit interval, for any θ, we have that $\Delta_\mu^-(\theta, q) = D_\mu^\pm(q) = 1$ for any q, $\Delta_\mu^+(\theta, q) = 1$ for $q \ge 0$ and $\Delta_\mu^+(\theta, q) = \infty$ for $q < 0$. Therefore, in this case, measure and return time generalized dimensions $D_\mu^\pm(q)$ and $D_\tau^\pm(q)$ coincide for all $q \le 2$. At the same time, for $q < 0$, $\Delta_\tau^+(\theta, q)$ and $\Delta_\mu^+(\theta, q)$ are affected by the "edge effect" discussed above and feature a "pathological" value.

As a consequence of the short-returns phenomenon discussed earlier in this section, Lemma 7, $D_\mu^\pm(q)$ and $D_\tau^\pm(q)$ differ for $q > 2$, and the latter dimensions vanish for large q. Observe also the "phase transition" behavior occurring at $q = 2$. Consider finally that, by choosing particular grids, we can obtain equality also for grid dimensions, when taking the superior limit (for $q > 0$) and the inferior limit in the opposite case.

We conjecture that what observed for this map is a rather common situation: that is to say, we expect that

Conjecture 1 For a large class of ergodic dynamical systems $D_\tau^\pm(q) = D_\mu^\pm(q)$ for $q_c < q \le 2$ (q_c being the lowest value of q for which partition functions of return times are finite, recall Lemma 4 and see below for an example) and $D_\tau^\pm(q) = D_\mu^\pm(2)/(q-1)$ for $q \ge 2$ (exactly, or at least asymptotically for large q).

At this point, it is relevant to quote the results of [6] that have already been mentioned in Sect. 1. They hold under strong assumptions on the dynamical system under investigation and for cylinders rather than balls (i.e. without relation to the geometric structure implied by the distance function). In fact, it has been shown that for Bowen–Gibbs measures, defining partition functions and generalized dimensions for entrance times (rather than return, see the remark about Grassberger technique in Sect. 2) in dynamical cylinders, these latter coincide with Renyi entropies for $q < 2$ and behave as $P(2\phi)/(q-1)$ for larger q. Here P is the topological pressure of the potential ϕ defining the Bowen–Gibbs measure. For return times in cylinders, though, only the statement for $q < 1$ has been derived. These results outline interesting techniques that might possibly be improved, and complemented with geometric considerations, to prove in vast generality the relations between measure and return time dimensions for balls, as originally conjectured in [18] and formulated above in a more precise form.

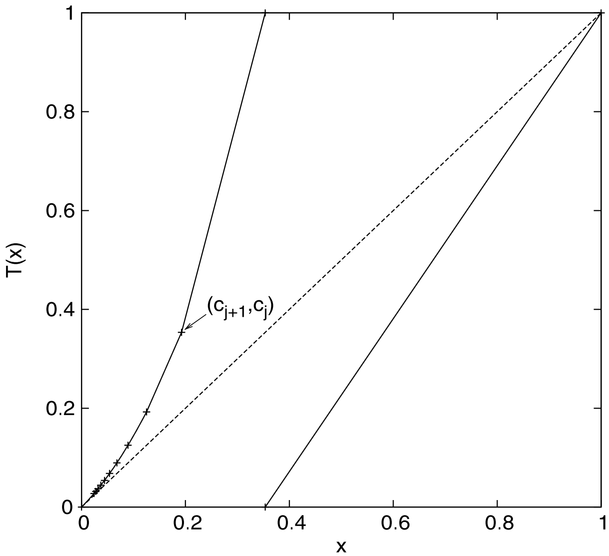


Fig. 2 Graph of the Gaspard–Wang map defined in (107), for $p = -3/2$, see Sect. 13 for details. Also drawn is the slope-one line through the origin

Let us now move to a final example, which shows that inequalities between measure and return time dimensions may be strict also for negative values of q: in fact, $D^{\pm}_{\tau}(q)$ and $\Delta^{\pm}_{\tau}(q)$ may be infinite for all q smaller than a critical value $q_c < 0$, while on the contrary $D^{\pm}_{\mu}(q)$ is finite. This is notably the case of intermittent maps, the simplest of which is perhaps the Gaspard–Wang [9] piece-wise linear approximation of the Pomeau–Manneville map [30] described in Sect. 13 and pictured in Fig. 2. This is a map of the unit interval into itself, with an absolutely continuous invariant measure. Zero is a fixed point of the map and the dynamics may spend arbitrarily long time spans in its neighbourhood. For this dynamical system we can prove the following theorem, that demonstrates a case where $D^{\pm}_{\tau}(q) > D^{\pm}_{\mu}(q)$ for sufficiently negative q, an inequality that is specific to this particular case and is not included among those in formulae (58) and (59).

Theorem 5 *In the dynamical system* $(T, [0,1], \mu)$, *where* T *is the Gaspard–Wang map defined in* (107) *with parameter* $p < -1$ *and* μ *is its unique absolutely continuous invariant measure*, *return time dimensions satisfy* $D^{\pm}_{\tau}(q) = \Delta^{\pm}_{\tau}(q) = \infty$ *for all* $q < q_c := p + 1$, *while measure dimensions take the values* $D^{\pm}_{\mu}(q) = 1$ *for* $q \leq -p$ *and* $D^{\pm}_{\mu}(q) = (1 + \frac{1}{p})\frac{q}{q-1}$ *for* $q \geq -p$.

Proof Is detailed in Sect. 13. It makes use of the partition function presented in Theorem 2. □

10 Conclusions

We can now conclude by saying that the idea to use return times in a straightforward way to compute generalized measure dimensions, following the programme whose history has been briefly outlined in the Introduction, is only applicable after a detailed analysis of the dynamical system considered.

The general inequalities that we have derived clarify the mutual relations among the dimensions that we have defined. As a by-product, these inequalities provide *universal bounds* for the global statistics of return times that hold for *all* ergodic dynamical systems possessing a given invariant measure μ, and indeed also for a large class of ergodic stochastic processes having invariant distribution μ. Remark in fact that the formalism developed in this paper also applies when T is a stochastic process, rather than a deterministic dynamical map.

We have found examples where $D_\mu(q)$ and $D_\tau(q)$ differ for $q > 2$, or for $q < q_c$. At the present moment, we do not know of any case where $D_\mu(q)$ and $D_\tau(q)$ are not equal in the interval $(q_c, 2)$. Nevertheless, we are not able to prove such equality in full generality with the means employed in this paper. We consider this, as well as the precise formulation and proof of Conjecture 1, to be a point of utmost interest for future investigations.

Turning from the general case to specific applications, we feel that one could prove part or all of Conjecture 1 with problem-specific tools. This might indeed be good news, that would partly fulfill the original Grassberger and Jensen et al. program. We look with particular interest to dimensions with negative q, that are known to be more elusive to compute numerically and more intriguing theoretically than those for positive q [10, 26].

Finally, whether linked to generalized measure dimensions or not, the moments of return times studied in this work deserve attention in their own, in our view. In fact, at difference with local quantities studied in the literature (such as probabilities of return to shrinking neighborhoods of a given point—a well examined topic, see e.g. [1, 17, 20, 25, 32]) they provide a global characteristic of the dynamics of a system.

The remainder of this paper consists of three sections giving details and proofs for the two maps quoted in this paper.

11 The Map of von Neumann and Kakutani

In this section we present the details of the intervals exchange map due to von Neumann and Kakutani [35], mentioned in Sect. 9. The basic properties of this map are known, but we prefer to re-derive them here for completeness and because they help us to understand some subtleties of return times for this map.

We start by defining two families of intervals in [0, 1]. The first is

$$J_k := [1 - 2^{-k}, 1 - 2^{-k-1}), \quad k = 0, 1, \ldots. \tag{67}$$

Clearly, $X = [0, 1] = \bigcup_{n=0}^{\infty} I_n \cup \{1\}$. Then, the map of von Neumann and Kakutani, T, is defined as follows:

$$T(x) := \begin{cases} x - 1 + 2^{-k} + 2^{-k-1} & \text{for } x \in J_k, \\ 0 & \text{for } x = 1. \end{cases} \tag{68}$$

The map T is piece-wise continuous, composed of an infinite number of affine segments and invertible (except for the point $x = 1$ which has no preimage).

In addition, for any positive integer n, define a measurable partition of X in (open) binary intervals:

$$A_j^n := (j2^{-n}, (j + 1)2^{-n}), \quad j = 0, \ldots, 2^n - 1. \tag{69}$$

All but a finite number of points in X are covered by the partition. Exception are the *boundary points* $\zeta_k^n = k2^{-n}$, with $k = 0, \ldots, 2^n$. Then, it is easy to see that

Lemma 8 *For any positive n, the map T permutes the family of intervals $\{A_j^n\}$. The permutation is cyclic, of period $N = 2^n$, in the in the sense that $T^N(A_j^n) = A_j^n$ for any j, and no shorter N exists with this property.*

Proof Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be the binary expansion of j, defined as follows (notice the order of digits):

$$j := \sum_{k=1}^{n} \sigma^k 2^{n-k}. \tag{70}$$

Let also $k(\sigma)$ be the index of the first zero in σ:

$$k(\sigma) := \min\{j \in \mathbf{N} \text{ s.t. } \sigma_j = 0\}. \tag{71}$$

Intervals A_j^n shall therefore be labelled as A_σ^n, where σ is a word of length n. We shall also use the complementary digit function $\bar{\cdot}$, where $\bar{0} = 1$, $\bar{1} = 0$.

All points in the interval A_σ^n can be written in binary form as $x = 0.\omega_1\ldots\omega_n\omega_{n+1}\ldots$, where $\omega_i = \sigma_i$ for $i = 1,\ldots,n$ and where $\omega_{n+1},\ldots$ is any infinite sequence of digits (except for the sequence composed of all ones). It can be verified that, in binary notation, the map T, (68), corresponds to the symbolic map $S(\omega) = .\eta_1\eta_2\ldots$, with

$$\eta_j := \begin{cases} \overline{\omega_j} & \text{for } j \le k(\omega), \\ \omega_j & \text{for } j > k(\omega). \end{cases} \tag{72}$$

Therefore, any interval A_σ^n is mapped into the interval A_η^n, labelled by the first n digits of η. For this reason, with a slight misusage of notation, we shall indicate by S also the map $\sigma \to \eta$ on the set of n-letter words, or equivalently via (70) on the set of integers $[0, 2^n - 1]$. The map S acts a cyclic permutation of all intervals A_σ^n, of period $N = 2^n$, for any value of n, the length of the word σ. □

An interesting consequence of the previous lemma is the following proposition,

Proposition 4 *The Lebesgue measure λ on X is invariant and ergodic for the action of the map T. Moreover, the dynamical system (X, T, λ) is metrically and topologically transitive, but not mixing.*

Proof The first statement is almost immediate from the form of the map T, (68) and the first part of Lemma 8: given any open interval I, its counter-image is a finite union of disjoint intervals whose lengths add up to the length of I.

To prove ergodicity one needs to show that for any measurable sets B and C,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \mu(T^{-k}(B) \cap C) = \mu(B)\mu(C). \tag{73}$$

Let B and C be finite unions of binary intervals A_j^n at resolution n. Indicate with $\mathcal{B}$ and $\mathcal{C}$ the sets of indices of the intervals composing the sets B and C, like in $B := \bigcup_{j\in\mathcal{B}} A_j^n$. Finally let $\#(\mathcal{B})$ and $\#(\mathcal{C})$ be the cardinalities of these sets, respectively. Recall that T permutes the intervals A_j^n as in A_j^n as in Lemma 8 and so does T^{-1}. Therefore,

$$T^{-k}(B) = T^{-k}\left(\bigcup_{j\in\mathcal{B}} A_j^n\right) = \bigcup_{j\in\mathcal{B}} T^{-k}(A_j^n), \tag{74}$$

and the intervals in the union above are disjoint, so that

$$\mu(T^{-k}(B) \cap C) = \sum_{j\in\mathcal{B}} \mu(T^{-k}(A_j^n) \cap C). \tag{75}$$

Moreover, for any j and k, either $T^{-k}(A^n_j)$ has empty intersection with C, or it coincides with one of the binary intervals composing C. It is then convenient to just consider the interval index map, that we have also indicated by T. Define therefore the set of "times" for which such intersection is not empty:

$$N^n_{\mathcal{C}}(j) := \{k \in \mathbf{Z} \text{ s.t. } 0 \leq k < 2^n \text{ and } T^{-k}(j) \in \mathcal{C}\}. \tag{76}$$

For each $k \in N^n_{\mathcal{C}}(j)$ we have that

$$\mu(T^{-k}(A^n_j) \cap C) = \mu(T^{-k}(A^n_j)) = \mu(A^n_j) = 2^{-n}, \tag{77}$$

and for $k \notin N^n_{\mathcal{C}}(j)$, $\mu(T^{-k}(A^n_j) \cap C) = 0$. We then compute

$$\begin{aligned}\sum_{k=0}^{2^n-1} \mu(T^{-k}(B) \cap C) &= \sum_{j \in \mathcal{B}} \sum_{k=0}^{2^n-1} \mu(T^{-k}(A^n_j) \cap C) \\ &= \sum_{j \in \mathcal{B}} \sum_{k \in N^n_{\mathcal{C}}(j)} \mu(A^n_j) = 2^{-n} \sum_{j \in \mathcal{B}} \# N^n_{\mathcal{C}}(j).\end{aligned} \tag{78}$$

Finally, observe that being T a cyclic permutation of the first 2^n integers, $T^{-k}(j) \in \mathcal{C}$ holds $\#(\mathcal{C})$ times along any cycle of times of length 2^n:

$$\# N^n_{\mathcal{C}}(j) = \#(\mathcal{C}), \tag{79}$$

which means

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} \mu(T^{-k}(B) \cap C) = 2^{-2n} \#(\mathcal{C}) \#(\mathcal{B}) = \mu(B)\mu(C). \tag{80}$$

This easily entails the limit (73) for binary intervals. Since these latter generate the Borel sigma algebra, the result follows generally.

Topological transitivity (ergodicity) is easily implied by Lemma 8, since given any two open sets B and C there exist an $n > 0$ and A^n_j, $A^n_{j'}$, $0 \leq j, j' < 2^n$, such that $A^n_j \subset B$ and $A^n_{j'} \subset C$. Choose then k such that $T^k(j) = j'$ to obtain the result.

It is also immediate to see that strong mixing is not present: in fact, this is ruled out by the cyclic nature of the images $T^{-k}(A^n_j)$, for a single n, j.

Weak mixing can be ruled out by a careful usage of (80). $\square$

The interesting properties of the map T so defined permit us to prove the following

Proposition 5 *In the dynamical system (T, λ) defined in this section, over the sequence $\varepsilon_n = 2^{-n}$, the return time partition function $\Psi_\tau(\theta, \varepsilon, q)$ can be explicitly computed when θ is the grid having origin at zero.*

Proof Let $\varepsilon_n = 2^{-n}$ and consider the newly introduced partition function $\Psi_\tau(\theta, \varepsilon, q)$, (35). It requires the computation of $\tau_{\overline{A^n_j}}(x)$ for $x \in A^n_j$. This we shall do now.

Observe first that $\overline{A^n_j} = \bigcup_{i=-1,0,1} A^n_{j+i}$, where obviously $A^n_j := \emptyset$ for $j < 0$ or $j \geq 2^n$. Because of the Lemma 8, in the von Neumann–Kakutani map T, these times are independent of the point x in A^n_j and can be computed in terms only of the index map S. Let $j = j(x)$ the index of the binary interval containing x. Then,

$$\tau_{\overline{A^n_j}}(x) = \inf\{k \geq 1 \text{ s.t. } S^k(j) \in \{j-1, j, j+1\}\}. \tag{81}$$

It is now possible (although complicated) to compute *explicitly* the distribution of first return times of the index j into $\{j-1, j, j+1\}$. An example will be provided in the next section. Observe that these return times over a finite set can take any value between one and the cardinality of the finite set, 2^n. Let ϱ^n_m the cardinality of the set of values of j for which the return value is m. Then, the following formula holds:

$$\varrho^n_m = \begin{cases} 2^l & \text{for } m = 3 \times 2^l,\ l = 0, \ldots, n-3, \\ 2^{n-2} & \text{for } m = 2^{n-2}, \\ 2^{n-1}+1 & \text{for } m = 2^{n-1}, \\ 0 & \text{elsewhere.} \end{cases} \tag{82}$$

Therefore, letting

$$\rho^{(n)}(m) := \mu(\{x \in X \text{ s.t. } \tau_{\overline{A^n_j}}(x) = m\}) = 2^{-n}\varrho^n_m \tag{83}$$

we can define a family of distribution functions over the integers larger than, or equal to one, according to which

$$\Psi_\tau(\theta, \varepsilon, q) := \sum_j \int_{A_j} \tau^{1-q}_{\overline{A_j}}(x)\, d\mu(x) = \sum_{m=1}^{\infty} m^{1-q} \rho^{(n)}(m), \tag{84}$$

with $\varepsilon = 2^{-n}$. □

12 Proof of Theorem 4 on the Map of von Neumann and Kakutani

Consider the measurable partitions $\{A^n_j\}$ of X in binary intervals of length 2^{-n} defined in (69) in Sect. 11 and let A^n_x be the element of the partition containing the point x. We call n the order of the partition. Lemma 8 implies that

$$\tau_{A^n_x}(x) = 2^n, \tag{85}$$

for any integer n and for any $x \in X$. We part the proof of Theorem 4 in several sections.

12.1 Part A, Where Large Balls Completely Cover Dyadic Intervals

In fact, when $2^{-n} \leq \varepsilon \leq 2^{-n+1}$ the ball of radius ε centered at x covers the dyadic interval including x: $B_\varepsilon(x) \supset A^n_x$. Therefore,

$$\tau_{B_\varepsilon(x)}(x) \leq \tau_{A^n_x}(x) = 2^n. \tag{86}$$

Choose q such that $1 - q \gtrless 0$. We now derive inequalities bounding the partition function $\Gamma_\tau(\varepsilon, q)$ in both cases. Firstly,

$$\int [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) \lesseqgtr \int [\tau_{A^n_x}(x)]^{1-q} d\mu(x) = 2^{n(1-q)}. \tag{87}$$

Observe that substituting the inequalities linking ε and n one gets

$$\log \Gamma_\tau(\varepsilon, q) \leq (q-1)\log\varepsilon \tag{88}$$

for $1-q>0$ and

$$\log \Gamma_\tau(\varepsilon, q) \geq (q-1)(\log\varepsilon + \log 2) \tag{89}$$

for $1-q<0$. This implies that

$$D_\tau^{\pm}(q) \leq 1 \tag{90}$$

for all values of $q \neq 1$ (a similar treatment could also yield the case $q=1$, we do not include it here for conciseness).

12.2 Part B, Where We Exploit Balls Included in Dyadic Intervals of Order n

Let ε and n be related as:

$$\frac{1}{8}2^{-n} \leq \varepsilon \leq \frac{1}{4}2^{-n}. \tag{91}$$

Then, when x is close to the midpoint of the interval A_x^n, $B_\varepsilon(x) \subset A_x^n$, so that

$$\tau_{B_\varepsilon(x)}(x) \geq \tau_{A_x^n}(x) = 2^n. \tag{92}$$

The same inequality also holds for points close to zero and one, the extrema of X, because in this case $(X \cap B_\varepsilon(x)) \subset A_x^n$. Collectively, all these points define the set $G_{n,\varepsilon}$. It is easy to see that because of (91) the measure of this set amounts to at least half of the total measure. We now take $1-q>0$, so that

$$\begin{aligned}\int_X [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) &\geq \int_{G_{n,\varepsilon}} [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) \\ &\geq \int_{G_{n,\varepsilon}} [\tau_{A_x^n}(x)]^{1-q} d\mu(x) \geq \frac{1}{2}2^{n(1-q)}.\end{aligned} \tag{93}$$

Proceeding as above, we find that

$$D_\tau^{\pm}(q) \geq 1 \tag{94}$$

for all values of $q \leq 1$. Together with (90) this implies that

$$D_\tau^{\pm}(q) = 1 \tag{95}$$

for all values of $q<1$.

12.3 Part C, Where We Exploit Balls Included in Dyadic Intervals of Orders 0 to n

Let the inequalities (91) still hold. We extend the argument of part B. Suppose that x does not belong to $G_{n,\varepsilon}$. This means that x is within ε of any of the endpoints of the interval A_x^n internal to $[0,1]$. Then, the ball $B_\varepsilon(x)$ is not included in A_x^n, but it stretches to reach one neighboring element of the partition.

Forcefully, $B_\varepsilon(x)$ includes a boundary point of the measurable partition of order n, of the form $\zeta_k^n = k2^{-n}$, with k integer. Clearly, two cases are possible: either ζ_k^n is an "even" point

(k even, in which case it is also a boundary point of the partition of order $n-1$), or it is an "odd" point. If it is an odd point, $B_\varepsilon(x)$ is necessarily included in A_x^{n-1} and $\tau_{B_\varepsilon(x)}(x) \geq \tau_{A_x^{n-1}}(x) = 2^{n-1}$. Let $G_{\varepsilon,n-1}$ be the set of points x that are ε-close to odd boundary points of the partition of order n. It is immediate that this set is composed of 2^{n-1} intervals of length 2ε.

The construction can be clearly iterated, by considering among even boundary points of level n those which are odd at level $n-1$: this defines a new set of points $G_{\varepsilon,n-2}$ consisting of 2^{n-2} intervals of length 2ε. For any x belonging to this set, $\tau_{B_\varepsilon(x)}(x) \geq \tau_{A_x^{n-2}}(x) = 2^{n-2}$ and so on. The last stage of the construction is the set $G_{\varepsilon,0} = (\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$, for which, rather trivially, $\tau_{B_\varepsilon(x)}(x) \geq \tau_{A_x^0}(x) = 2^0 = 1$.

The above proves the following formulæ: for any $\varepsilon > 0$ and n satisfying the inequalities (91), one has:

$$X = \bigcup_{j=0}^{n} G_{\varepsilon,j}, \tag{96}$$

with $\lambda(G_{\varepsilon,j} \cap G_{\varepsilon,j'}) = 0$ if $j \neq j'$ (recall that λ is the Lebesgue measure and that $G_{\varepsilon,n}$ has been defined in point b), so to provide another measurable partition of X. Moreover,

$$2^{-n+j} \geq 2^{-n-1+j} \geq \lambda(G_{\varepsilon,j}) \geq 2^{-n-2+j}, \tag{97}$$

for $0 \leq j \leq n-1$ and $1 \geq \frac{3}{4} + 2^{-n-2} \geq \lambda(G_{\varepsilon,n}) \geq 2^{-1}$. Finally,

$$\tau_{B_\varepsilon(x)}(x) \geq 2^j \tag{98}$$

for any $x \in G_{\varepsilon,j}$ and $j = 0, \ldots, n$.

We can now evaluate the partition function: let $1-q<0$, i.e. $q>1$, so that

$$\Gamma_\tau(\varepsilon,q) = \sum_{j=0}^{n} \int_{G_{\varepsilon,j}} [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) \leq \sum_{j=0}^{n} \lambda(G_{\varepsilon,j}) 2^{j(1-q)} \leq \sum_{j=0}^{n} 2^{-n+j} 2^{j(1-q)}, \tag{99}$$

where we have used the widest inequality in (97) to obtain a simpler formula. In fact, (99) easily yields

$$\Gamma_\tau(\varepsilon,q) \leq G(q,n) := 2^{-n} \frac{2^{(2-q)(n+1)}-1}{2^{(2-q)}-1}, \tag{100}$$

where the function $G(q,n)$ has been defined. Two cases must now be considered in the asymptotics of $G(q,n)$ as n tends to infinity, or equivalently ε goes to zero, according to the inequalities (91). First, when $2>q>1$ we find $\log(G(q,n)) \sim \varepsilon^{q-1}$, so that $D_\tau^\pm(q) \geq 1$ and finally

$$D_\tau^\pm(q) = 1 \tag{101}$$

for all values of $q \leq 2$ in the other case, $q>2$, we find $\log(G(q,n)) \sim \varepsilon^{-1}$, so that

$$D_\tau^\pm(q) \geq \frac{1}{q-1}. \tag{102}$$

12.4 Part D, Where We Exploit the Return Properties of a Particular Sequence of Dyadic Intervals

This is the last part of this proof. In Proposition 5 we have computed the return times of dyadic intervals in their enlarged neighborhood, according to Sects. 5, 6. We need a particular case of Proposition 5 that can be easily proven. It appears from (82) that *for any* n there exist *one* interval A^n_j with return time 3 inside $\overline{A}^n_j$. The index of this interval is $j = 2^{n-1} - 1$ and its symbolic address is $01\dots1$. Let this value of j be fixed in the following. It maps to $11\dots1$ and successively $0\dots0$ and $10\dots0$. It is easy to see that the first and the last are neighboring intervals. In coordinates, the quasi-cycle is $A^n_j = (\frac{1}{2} - 2^{-n}, \frac{1}{2}) \to (1 - 2^{-n}, 1) \to (0, 2^{-n}) \to (\frac{1}{2}, \frac{1}{2} + 2^{-n}) = A^n_{j+1}$. It contains the orbit $1 \to 0 \to \frac{1}{2}$. The periodic sequence $\frac{1}{2} \to 1 \to 0 \to \frac{1}{2}$ does *not* belong to any orbit, but it is arbitrarily well approximated by true orbits.

If we now take $2^{-n+2} \geq \varepsilon \geq 2^{-n+1}$ we have that $B_\varepsilon(x) \supset A^n_{j+1}$ for any x in A^n_j and the above implies that $\tau_{B_\varepsilon(x)}(x) \leq 3$ on A^n_j. Therefore, when $1 - q < 0$,

$$\int_X [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) \geq \int_{A^n_j} [\tau_{B_\varepsilon(x)}(x)]^{1-q} d\mu(x) \geq 3^{1-q} 2^{-n}. \tag{103}$$

This yields $D^\pm_\tau(q) \leq \frac{1}{q-1}$ for any $q > 1$ and finally

$$D^\pm_\tau(q) = \frac{1}{q-1} \tag{104}$$

for any $q \geq 2$. This ends the proof of the part of the theorem concerning $D^\pm_\tau(q)$.

12.5 Proof of the Results for Box Dimensions

As for the box dimension $\Delta^\pm(\theta, q)$, choose now the grid θ with origin at zero and consider the sequence $\varepsilon_n = 2^{-n}$. Recall (85). It implies that $\Upsilon_\tau(\theta, \varepsilon_n, q) = 2^{n(1-q)} = \varepsilon_n^{q-1}$. Then, for all $q \neq 1$

$$\lim_{n\to\infty} \frac{1}{q-1} \frac{\log \Upsilon_\tau(\theta, \varepsilon_n, q)}{\log \varepsilon_n} = 1. \tag{105}$$

Let now $q < 0$. Clearly,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{q-1} \frac{\log \Upsilon_\tau(\theta, \varepsilon_n, q)}{\log \varepsilon_n} &\geq \liminf_{\varepsilon\to 0} \frac{1}{q-1} \frac{\log \Upsilon_\tau(\theta, \varepsilon, q)}{\log \varepsilon} \\ &:= \Delta^-_\tau(\theta, q) \geq \Delta^-_\mu(\theta, q) = 1, \end{aligned} \tag{106}$$

where the second inequality is (59) and where the last equality can be easily obtained. Since the first limit exists and is equal to one, (105), so is $\Delta^-_\tau(\theta, q)$. For negative q a similar argument applies, which now requires the superior limit. The same results are obviously found when the origin of the grid θ is a point of the form $k\,2^{-m}$, with integer k and m. This ends the proof of the theorem.

13 The Map of Gaspard and Wang

We now describe the piece-wise linear approximation of the Pomeau Manneville intermittent map [30] due to Gaspard and Wang [9]. Let $\{c_j\}_{j\in\mathbf{N}}$ be an ordered, decreasing sequence of real numbers between zero and one such that $c_0 = 1$ and such that c_j tends to zero as j tends to infinity. Let $I_j := (c_{j+1}, c_j)$ be the elements of a partition of [0, 1] into open intervals of length $l_j = c_j - c_{j+1}$. The map T is defined as the transformation which maps affinely and with positive slope I_j onto I_{j-1} for $j \geq 1$ and I_0 onto [0, 1]:

$$T(x) = (x - c_{j+1})\frac{l_{j-1}}{l_j} + d_j, \tag{107}$$

for $x \in I_j$ and where we set $l_{-1} = 1$ and $d_j = c_j$ for $j > 0$, $d_0 = 0$. Because this behavior, it is also called an *infinite renewal chain* map. We study in this work the family of such maps, parameterized by $p < -1$, for which

$$c_j = (j + 1)^p. \tag{108}$$

Proof of Theorem 5 The absolutely continuous invariant measure on [0, 1], whose density is constant on each I_j can be easily computed. One finds

$$\mu(I_j) = c_j a \tag{109}$$

where the parameter $a = \mu(I_0)$ can be chosen so to normalize the measure, of course when the sequence $\{c_j\}_{j\in\mathbf{N}}$ is summable, which is always the case when $p < 1$. The motion of this dynamical system is such that I_0 can be also parted into an infinity of adjacent intervals K_j, $j = 0, \ldots,$ whose points return to I_0 after exactly $j + 1$ steps: if $x \in K_j$, $\tau_{I_0}(x) = j + 1$. One finds easily that

$$T(K_j) = I_j, \tag{110}$$

so that all K_j can be obtained by an affine transformation of I_j: $K_j = (1 - c_1)I_j + 1$. The measure of K_j is proportional to its length and hence to the length of I_j:

$$\mu(K_j) = al_j. \tag{111}$$

Therefore, not all moments of the return times of points of I_0 into itself are finite: in fact,

$$\int_{I_0} \tau_{I_0}^{1-q}(x)d\mu(x) = \sum_{j=0}^{\infty}\int_{K_j} \tau_{I_0}^{1-q}(x)d\mu(x) = a\sum_{j=0}^{\infty}(j + 1)^{1-q}l_j. \tag{112}$$

Since $l_j \sim j^{p-1}$, the above integral is convergent only when $q > p + 1$. Also observe that formulae (51) are here strict inequalities.

Now, let us cover the unit interval by a box grid of side ε and let's evaluate the partition function $\Psi_\tau(\theta, \varepsilon, q)$. Let us consider the particular box A that contains the point c_1 in its interior. If c_1 is a boundary point, consider the box whose left extremum is c_1. For any $\varepsilon > 0$ the box A contains an infinite number of K_j, those with $j > j_\varepsilon$. At the same time, the enlarged box $\overline{A}$ is enclosed in the union $I_0 \cup I_1 \cup \cdots \cup I_m$, when m depends on ε. If ε is sufficiently small, we can take $m = 1$. Since K_j maps to I_j and this to I_{j-1}, et cetera, the time to return to $\overline{A}$ is larger that the time to enter I_m from K_j. This latter is, clearly, $j-m$. Entering this information in (35) we find that, when $q<1$,

$$\Psi_\tau(\theta,\varepsilon,q) \geq \int_A \tau_{\overline{A}}^{1-q}(x)d\mu(x) \geq \sum_{j>j_\varepsilon} \mu(K_j)(j-m)^{1-q}. \tag{113}$$

Therefore, as in (112), the sum diverges for $q<p+1$ and $D_\tau^\pm(q)=\infty$ for these values. Because of (59) the same happens for $\Delta_\tau^\pm(q)$.

Finally, since the density of μ over I_j grows as j, which is the same as $x^{\frac{1}{p}}$, standard theory gives the formula for the generalized dimensions $D_\mu(q)$. $\square$

Notice that divergence of certain moments of entrance and return times in cylinders for the Manneville–Pomeau map has been exhibited in [6]. Further details on this map, and a *local* analysis of return and entrance times can be found in [5] in the case when the invariant measure is infinite.

References

1. Abadi, M.: Sharp error terms and necessary conditions for exponential hitting times in mixing processes. Ann. Probab. **32**, 243–264 (2004)
2. Badii, R., Politi, A.: Statistical description of chaotic attractors: the dimension function. J. Stat. Phys. **40**, 725–750 (1985)
3. Badii, R., Politi, A.: Renyi dimensions from local expansion rates. Phys. Rev. A **35**, 1288–1293 (1987)
4. Barbaroux, J.M., Germinet, F., Tcheremchantsev, S.: Generalized fractal dimensions: equivalence and basic properties. J. Math. Pures Appl. **80**, 977–1012 (2001)
5. Bressaud, X., Zweimüller, R.: Non exponential law of entrance times in asymptotically rare evens for intermittent maps with infinite invariant measure. Ann. Henri Poincaré **2**, 501–512 (2001)
6. Chazottes, J.-R., Ugalde, E.: Entropy estimation and fluctuations of hitting and recurrence times for Gibbsian sources. Discrete Contin. Dyn. Syst. **5**, 565–586 (2005)
7. Collet, P., Galves, A., Schmitt, B.: Fluctuations of repetition times for Gibbsian sources. Nonlinearity **12**, 1225–1237 (1999)
8. Cutler, C.D.: Some results on the behaviour and estimation of the fractal dimensions of distributions on attractors. J. Stat. Phys. **62**, 651–708 (1991)
9. Gaspard, P., Wang, X.-J.: Sporadicity: between periodic and chaotic dynamical behaviors. Proc. Natl. Acad. Sci. USA **85**, 4591–4595 (1988)
10. Germinet, F., Tcheremchantsev, S.: Generalized fractal dimensions on the negative axis for compactly supported measures. Math. Nachr. **279**, 543–570 (2006)
11. Grassberger, P.: Generalized dimension of strange attractors. Phys. Lett. A **97**, 227–230 (1983)
12. Grassberger, P., Procaccia, I.: Characterization of strange sets. Phys. Rev. Lett. **50**, 346–349 (1983)
13. Gratrix, S., Elgin, J.N.: Pointwise dimension of the Lorenz Attractor. Phys. Rev. Lett. **92**, 14101 (2004)
14. Guysinsky, M., Yaskolko, S.: Coincidence of various dimensions associated with metrics and measures on metric spaces. Discrete Contin. Dyn. Syst. **3**, 591–603 (1997)
15. Halsey, T.C., Jensen, M.H.: Hurricanes and butterflies. Nature **428**, 127 (2004)
16. Halsey, T.C., Jensen, M., Kadanoff, L., Procaccia, I., Shraiman, B.: Fractal measures and their singularities: the characterization of strange sets. Phys. Rev. A **33**, 1141–1151 (1986)
17. Haydn, N., Vaienti, S.: The limiting distribution and error terms for return times of dynamical systems. Discrete Contin. Dyn. Syst. **10**, 589–616 (2004)
18. Haydn, N., Luevano, J., Mantica, G., Vaienti, S.: Multifractal properties of return time statistics. Phys. Rev. Lett. **88**, 224502 (2003)
19. Hentschel, H.G.E., Procaccia, I.: The infinite number of generalized dimensions of fractals and strange attractors. Physica D **8**, 435–444 (1983)
20. Hirata, M., Saussol, B., Vaienti, S.: Statistics of return times: a general framework and new applications. Commun. Math. Phys. **206**, 33–55 (1999)
21. Jensen, M.H., Kadanoff, L.P., Libchaber, A., Procaccia, I., Stavans, J.: Global universality at the onset of chaos: results of a forced Rayleigh–Bénard experiment. Phys. Rev. Lett. **55**, 2798–2801 (1985)

22. Kac, M.: Probability and Related Topics in Physical Sciences. AMS, Providence (1959)
23. Luevano, J., Penne, V., Vaienti, S.: Multifractal spectrum via return times. Preprint mp-arc 00-232 (2000)
24. Olsen, L.: First return times: multifractal spectra and divergence points. Discrete Contin. Dyn. Syst. **10**, 635–656 (2004)
25. Ornstein, D.S., Weiss, B.: Entropy and data compression schemes. IEEE Trans. Inf. Theory **39**, 78–83 (1993)
26. Pastor-Satorras, R., Riedi, R.H.: Numerical estimates of the generalized dimensions of the Hénon attractor for negative q. J. Phys. A, Math. Gen. **29**, L391–L398 (1996)
27. Pesin, Ya.: Dimension Theory in Dynamical Systems. University of Chicago Press, Chicago (1997)
28. Pesin, Ya., Weiss, H.: A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions. J. Stat. Phys. **86**, 233–275 (1997)
29. Petersen, K.E.: Ergodic Theory. Cambridge University Press, Cambridge (1989)
30. Pomeau, Y., Manneville, P.: Intermittent transition to turbulence in dissipative dynamical systems. Commun. Math. Phys. **74**, 189–197 (1980)
31. Riedi, R.H.: An improved multifractal formalism and self-similar measures. J. Math. Anal. Appl. **189**, 462–490 (1995)
32. Saussol, B.: On fluctuations and the exponential statistics of return times. Nonlinearity **14**, 179–191 (2001)
33. Schulz-Baldez, H., Bellissard, J.: Anomalous transport: a mathematical framework. Rev. Math. Phys. **10**, 1–46 (1998)
34. Vaienti, S.: Generalised spectra for the dimensions of strange sets. J. Phys. A **21**, 2313–2320 (1988)
35. von Neumann, J.: Zür Operatorenmethode in der klassichen Mechanik. Ann. Math. **33**, 587–642 (1932) (Our map is presented on page 630)